

# Compound and scale mixture of vector and spherical matrix variate elliptical distributions

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## Abstract

Several matrix variate hypergeometric type distributions are derived. The compound distributions of left-spherical matrix variate elliptical distributions and inverted hypergeometric type distributions with matrix arguments are then proposed. The scale mixture of left-spherical matrix variate elliptical distributions and uni-variate inverted hypergeometric type distributions is also derived as a particular case of the compound distribution approach.

## 1 Introduction

Four classes of matrix variate elliptical distributions have been defined and studied by Fang and Zhang (1990). That  $m \times n$  random matrix variate  $\mathbf{X}$  is said to have a matrix variate left-spherical distribution, the largest of the four classes class of matrix variate elliptical distributions, if its density function is given by

$$\frac{c(m, n)}{|\Sigma|^{n/2} |\Theta|^{m/2}} h \left( \Sigma^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \Theta^{-1} (\mathbf{X} - \boldsymbol{\mu}) \Sigma^{-1/2} \right),$$

where  $h$  is a real function,  $c(m, n)$  denotes the normalization constant,  $\Sigma$  is an  $m \times m$  positive definite matrix, this fact being denoted as  $\Sigma > \mathbf{0}$ ,  $\Theta$  is an  $n \times n$  matrix,  $\Theta > \mathbf{0}$ , and  $\boldsymbol{\mu}$  is an  $m \times n$  matrix.

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When implementing frequentist or Bayesian methods, one may be interested in situations where  $\mathbf{X}$  has a density function of the following form, see Fang and Zhang (1990) and Fang and Li (1999),

$$\frac{c(m, n)}{|\Sigma|^{n/2} |\Theta|^{m/2}} h(\Sigma^{-1}(\mathbf{X} - \mu)' \Theta^{-1}(\mathbf{X} - \mu)). \quad (1)$$

This fact is denoted as  $\mathbf{X} \sim \mathcal{EL}\mathcal{S}_{m \times n}(\mu, \Sigma, \Theta, h)$ . This condition is equivalent to considering the function  $h$  as a symmetric function, i.e.  $g : g(\mathbf{AB}) = g(\mathbf{BA})$  for any symmetric matrices  $\mathbf{A}$  and  $\mathbf{B}$ . This condition is equivalent to that in which  $h(\mathbf{A})$  depends on  $\mathbf{A}$  only through its eigenvalues, in which case the function  $h(\mathbf{A})$  can be expressed as  $h(\lambda(\mathbf{A}))$ , where  $\lambda(\mathbf{A}) = \text{diag}(\lambda_1, \dots, \lambda_m)$  and  $\lambda_1, \dots, \lambda_m$  are the eigenvalues of  $\mathbf{A}$ . Two subclasses of matrix variate elliptical distributions are of particular interest: the vector and spherical matrix variate elliptical distributions. For these distributions,  $\lambda(\mathbf{A}) \equiv \text{tr}(\mathbf{A})$  and  $\lambda(\mathbf{A})$  represents any function of eigenvalues of  $\mathbf{A}$ , respectively. Note that vector matrix variate elliptical distributions are a subclass of matrix variate spherical elliptical distributions. Many well-know distributions are examples of these subclasses; one such is the matrix variate normal distribution. Other variants include vector matrix variate elliptical distributions e.g. Pearson type II, Pearson type VII, Kotz type, Bessel and Logistic, among many others, see Gupta and Varga (1993). Yet other are matrix variate spherical elliptical distributions, e.g. Pearson Type II, Pearson type VII and Kotz type among many others, see Fang and Li (1999).

In the vectorial case, Muirhead (1982, p. 33) proposed a means of generating a family of vector variate elliptical distributions from a normal distribution. In the general case, this idea has been extended to the matrix variate elliptical distributions by Gupta and Varga (1993, pp. 78-79 and Section 4.1). The situation in which the specific elliptical distribution is a matrix variate normal is studied in Gupta and Varga (1993, Chapter 4). Generically, distributions obtained by this procedure are termed scale mixture matrix variate normal or elliptical distributions.

Arsilan (2005) proposed the scale mixture of the vector variate Kotz type distribution, also termed the  $t$ -type or generalised  $t$  distribution. Díaz-García and Gutiérrez-Jáimez (2009) extend this idea to matrix variate vector and spherical Kotz type distributions using two approaches: scale mixture and compound matrix variate distributions.

These forms of obtaining vector or spherical matrix variate elliptical distributions are of particular interest from a Bayesian point of view (Jammalamadaka *et al.*, 1987; Fang and Li, 1999) and in the context of shape theory, see Caro-Lopera *et al.* (2008).

This paper introduces several families of matrix variate elliptical distributions. Section 2 gives some results on integration, using zonal polynomials. In terms of these results, various matrix variate hypergeometric type distributions are proposed, as particular cases. These include well-known distributions such as central and noncentral matrix variate inverted gamma (inverted Wishart) distributions and matrix variate central and noncentral beta type II distributions. In Section 3, assuming a hypergeometric type distribution for the matrix parameter in matrix variate normal and matrix variate  $T$  distributions, several families of matrix variate elliptical distributions are obtained using the compound matrix variate approach. Section 6 introduces the scale mixture of a matrix variate elliptical distribution, which is derived as a particular case of the compound matrix variate approach, from where, all the results given in Section 3 can be particularised to this case.

## 2 Preliminary results

Consider the following notation and definitions: the hypergeometric functions  ${}_pF_q$  with matrix arguments are defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{Y}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_{\kappa} \cdots (a_p)_{\kappa}}{(b_1)_{\kappa} \cdots (b_q)_{\kappa}} \frac{C_{\kappa}(\mathbf{Y})}{k!},$$

where  $\mathbf{Y}$  is a complex symmetric  $m \times m$  matrix,  $C_{\kappa}(\mathbf{Y})$  is the zonal polynomial of  $\mathbf{Y}$  of degree  $k$ ,  $\kappa = (k_1, \dots, k_m)$ ,  $k_1 \geq \dots \geq k_m \geq 0$ ,  $k_1 + \dots + k_m = k$  and  $a_1, \dots, a_p, b_1, \dots, b_q$  are real or complex constants,

$$(a)_{\kappa} = \prod_{i=1}^m (a - (i-1)/2)_{k_i},$$

with  $(x)_n = x(x+1) \cdots (x+n-1)$ ,  $(x)_0 = 1$ .

The multivariate gamma function is defined as

$$\Gamma_m[a] = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma[a - (i-1)/2],$$

and

$$\Gamma_m[a, \kappa] = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)/2],$$

where  $\Gamma_m[a, \kappa] = (a)_{\kappa} \Gamma_m[a]$  with  $\text{Re}(a) > (m-1)/2$ , see Khatri (1966) and Muirhead (1982). Similarly,

$$\begin{aligned} \Gamma_m[a, -\kappa] &= \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma[a - k_i - (m-i)/2] \\ &= \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma[a - k_{m+1-i} - (i-1)/2], \end{aligned}$$

where  $\text{Re}(a) > (m-1)/2 + k_1$ . From

$$(-x)_q = (-1)^q (x - q + 1)_q = \frac{(-1)^q \Gamma[x+1]}{\Gamma[x-q+1]},$$

we obtain that

$$\Gamma_m[a, -\kappa] = \frac{(-1)^k \Gamma_m[a]}{(-a + (m+1)/2)_{\kappa}}.$$

The multivariate beta function is defined as

$$\beta_m[a, b] = \int_{\mathbf{0} < \mathbf{Y} < \mathbf{I}_m} |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}_m - \mathbf{Y}|^{b-(m+1)/2} (d\mathbf{Y}) = \frac{\Gamma_m[a] \Gamma_m[b]}{\Gamma_m[a+b]},$$

where  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2$ , see Herz (1955, p. 480).

Many distributions in multivariate analysis can be expressed in a form involving hypergeometric functions with matrix arguments, as considered by Roux (1975), see also Press (1982, Section 6.6.3, pp. 170-171). These distributions contain as particular cases the central and noncentral gamma (Wishart) and matrix variate beta type I and II distributions and are termed matrix variate hypergeometric gamma (Wishart) type and matrix variate hypergeometric beta type I and II distributions. In particular, for matrix variate hypergeometric beta type II distributions, we obtain an alternative expression to the one given by Roux (1975), based on the following lemma from Khatri (1966).

**Lemma 2.1.** *If  $\mathbf{R}$  is any arbitrary complex symmetric  $m \times m$  matrix, then*

$$\int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}_m + \mathbf{Y}|^{-(a+b)} C_\kappa(\mathbf{Y}\mathbf{R}) = \frac{(a)_\kappa \beta_m[a, b]}{(-b + (m+1)/2)_\kappa} C_\kappa(-\mathbf{R}),$$

where  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2 + k_1$ .

*Proof.* As given in Khatri (1966).  $\square$

If  $\mathbf{Y} > \mathbf{0}$  has a matrix variate hypergeometric beta type II distribution, the argument of the hypergeometric function involved, in Roux's version, is  $(\mathbf{I}_m + \mathbf{Y})^{-1}$ , whereas in the version based on Khatri's lemma, its argument is  $\mathbf{Y}$ , as we see below. The importance of this fact is made apparent in the next section.

The next result is obtained immediately from Lemma 2.1.

**Corollary 2.1.** *Let  $\mathbf{R}$  be any arbitrary complex symmetric  $m \times m$  matrix, then*

$$\begin{aligned} \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}_m + \mathbf{Y}|^{-(a+b)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{R}\mathbf{Y}) \\ = \beta_m[a, b] {}_{p+1}F_{q+1}(a_1, \dots, a_p, a; b_1, \dots, b_q, -b + (m+1)/2; -\mathbf{R}), \end{aligned}$$

where  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2 + k_1$ .

*Proof.* The final results are obtained using the fact that  $C_\kappa(b\mathbf{Y}) = b^\kappa C_\kappa(\mathbf{Y})$ , for a constant  $b$ .  $\square$

As a consequence of Corollary 2.1 we have the following alternative definition of the matrix variate hypergeometric beta type II distributions.

**Definition 2.1.** Let  $\Xi$  be any arbitrary complex symmetric  $m \times m$  matrix. Then  $\mathbf{Y}$  has a **matrix variate hypergeometric beta type II distribution** if its density function is

$$f_{\mathbf{Y}}(\mathbf{Y}) \propto |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}_m + \mathbf{Y}|^{-(a+b)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \Xi\mathbf{Y}), \quad \mathbf{Y} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{1}{\beta_m[a, b] {}_{p+1}F_{q+1}(a_1, \dots, a_p, a; b_1, \dots, b_q, -b + (m+1)/2; -\Xi)},$$

with  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2 + k_1$ .

Let  $\mathbf{Y}$  be a positive definite  $m \times m$  matrix and let us define  $\mathbf{P} = \mathbf{Y}^{-1}$  then  $(d\mathbf{Y}) = |\mathbf{P}|^{-(m+1)}(d\mathbf{P})$ . From Roux (1975, eq. (4.1)), we have the following result.

**Lemma 2.2.** *Let  $\Upsilon$  and  $\Xi$  be complex symmetric  $m \times m$  matrices with  $\text{Re}(\Xi) > \mathbf{0}$ . And assume that  $\mathbf{Y}$  has a hypergeometric matrix variate gamma type distribution. Then  $\mathbf{P} = \mathbf{Y}^{-1}$  has a **matrix variate inverted hypergeometric gamma type distribution** with the following density function:*

$$f_{\mathbf{P}}(\mathbf{P}) \propto \text{etr}\{-\Xi\mathbf{P}^{-1}\} |\mathbf{P}|^{-a-(m+1)/2} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \Upsilon\mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{|\Xi|^a}{\Gamma_m[a] {}_{p+1}F_q(a_1, \dots, a_p, a; b_1, \dots, b_q; \Upsilon \Xi^{-1})},$$

and  $\text{Re}(a) > (m-1)/2$ .

Observe that if in Lemma 2.2,  $\Psi = \mathbf{0}$ , then  $\mathbf{P}$  has a matrix variate central inverted gamma distribution. And if  $p = 0$ ,  $q = 1$  and  $a = b_1$ , then  $\mathbf{P}$  has a matrix variate noncentral inverted Gamma distribution.

Similarly, from Definition 2.1, we have

**Lemma 2.3.** *Let  $\Xi$  be any arbitrary complex symmetric  $m \times m$  matrix. Then  $\mathbf{P} = \mathbf{Y}^{-1}$  has a **matrix variate inverted hypergeometric beta type II distribution** and its density function is*

$$f_{\mathbf{P}}(\mathbf{P}) \propto |\mathbf{P}|^{b-(m+1)/2} |\mathbf{I}_m + \mathbf{P}|^{-(a+b)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \Xi \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{1}{\beta_m[a, b] {}_{p+1}F_{q+1}(a_1, \dots, a_p, a; b_1, \dots, b_q, -b + (m+1)/2; -\Xi)},$$

and  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2 + k_1$ .

The distribution in Lemma 2.3 contains as particular cases the matrix variate central and noncentral inverted beta type II distributions.

Van der Morwe and Roux (1974) studied other families of matrix variate hypergeometric distributions, based on the  ${}_2F_1(a, b; c; -\mathbf{Y})$  hypergeometric function with a matrix argument. From this, and taking the limits of parameters  $a$ ,  $b$  or  $c$ , they obtain the central and non-central matrix variate gamma distributions, the central matrix variate beta distribution and the matrix variate normal distribution. As we see in Mathai and Saxena (1966) many other well-known distributions can be obtained as particular cases of this distribution. Next, we introduce this distribution using the multivariate Mellin transform, Mathai (1997).

Let  $g(\mathbf{Y})$  be a function of the positive definite  $m \times m$  matrix  $\mathbf{Y}$ . The Mellin transform of  $g(\mathbf{Y})$  is defined as

$$M_g(\mathbf{Y}) = \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} g(\mathbf{Y}) (d\mathbf{Y}),$$

where  $\text{Re}(\alpha) > (m-1)/2$ .

**Lemma 2.4.** *The Mellin transform of  $g(\mathbf{Y}) = {}_2F_1(a, b; c; -\mathbf{Y})$  is given by*

$$\int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} {}_2F_1(a, b; c; -\mathbf{Y}) (d\mathbf{Y}) = \frac{\beta_m[\alpha, b-\alpha] \beta_m[a-\alpha, c-\alpha]}{\beta_m[a, c-a]},$$

where  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(a-\alpha) > (m-1)/2$ ,  $\text{Re}(b-\alpha) > (m-1)/2$ ,  $\text{Re}(c-\alpha) > (m-1)/2$  and  $\text{Re}(c-a) > (m-1)/2$ .

*Proof.* From the integral representation of  ${}_2F_1$ , see Herz (1955, eq. (2.12)) and Muirhead (1982, Theorem 7.4.2),

$$\begin{aligned} M_g(\mathbf{Y}) &= \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} {}_2F_1(a, b; c; -\mathbf{Y}) (d\mathbf{Y}) \\ &= \frac{1}{\beta_m[a, c-a]} \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_m} |\mathbf{R}|^{a-(m+1)/2} \\ &\quad \times |\mathbf{I}_m - \mathbf{R}|^{c-a-(m+1)/2} |\mathbf{I}_m - \mathbf{Y}\mathbf{R}|^{-b} (d\mathbf{R}) (d\mathbf{Y}) \\ &= \frac{1}{\beta_m[a, c-a]} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_m} |\mathbf{R}|^{a-(m+1)/2} |\mathbf{I}_m - \mathbf{R}|^{c-a-(m+1)/2} \\ &\quad \times \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} |\mathbf{I}_m - \mathbf{Y}\mathbf{R}|^{-b} (d\mathbf{Y}) (d\mathbf{R}). \end{aligned}$$

Note that,  $|\mathbf{I}_m - \mathbf{Y}\mathbf{R}| = |\mathbf{I}_m - \mathbf{R}^{1/2}\mathbf{Y}\mathbf{R}^{1/2}| = |\mathbf{I}_m - \mathbf{W}|$ , where  $\mathbf{R}^{1/2}$  is the positive definite square root of  $\mathbf{R}$ , such that  $\mathbf{R} = (\mathbf{R}^{1/2})^2$  (Muirhead, 1982, Theorem A9.3, p. 588) and  $|\mathbf{Y}| = |\mathbf{R}^{-1/2}\mathbf{W}\mathbf{R}^{-1/2}| = |\mathbf{W}||\mathbf{R}|^{-1}$ , with  $\mathbf{W} = \mathbf{R}^{1/2}\mathbf{Y}\mathbf{R}^{1/2}$ . Then,  $(d\mathbf{Y}) = |\mathbf{R}|^{-(m+1)/2}(d\mathbf{W})$ , from where

$$\begin{aligned} M_g(\mathbf{Y}) &= \frac{1}{\beta_m[a, c-a]} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_m} |\mathbf{R}|^{a-\alpha-(m+1)/2} |\mathbf{I}_m - \mathbf{R}|^{c-a-(m+1)/2} \\ &\quad \times \int_{\mathbf{W} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} |\mathbf{I}_m - \mathbf{W}|^{-b} (d\mathbf{W})(d\mathbf{Y}). \\ &= \frac{\beta_m[\alpha, b-\alpha]}{\beta_m[a, c-a]} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_m} |\mathbf{R}|^{a-\alpha-(m+1)/2} |\mathbf{I}_m - \mathbf{R}|^{c-a-(m+1)/2} (d\mathbf{Y}) \\ &= \frac{\beta_m[\alpha, b-\alpha]\beta_m[a-\alpha, c-\alpha]}{\beta_m[a, c-a]}. \quad \square \end{aligned}$$

This was proved by Van der Morwe and Roux (1974), who took the limit when  $a$  tends to infinity in Lemma 2.4 and found the Mellin transform of the function  $g(\mathbf{Y}) = {}_1F_1(b; c; -\mathbf{Y})$ . Alternatively, the results can be obtained directly by integration, as we shown below.

**Lemma 2.5.** *The Mellin transform of  $g(\mathbf{Y}) = {}_1F_1(b; c; -\mathbf{Y})$  is given by*

$$\int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} {}_1F_1(b; c; -\mathbf{Y})(d\mathbf{Y}) = \frac{\Gamma_m[\alpha]\Gamma_m[c]\Gamma_m[b-\alpha]}{\Gamma_m[b]\Gamma_m[c-\alpha]},$$

where  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(b-\alpha) > (m-1)/2$  and  $\text{Re}(c-\alpha) > (m-1)/2$ .

*Proof.* Noting that from Muirhead (1982, Theorem 7.4.2, p. 264)

$$\begin{aligned} {}_2F_1(c-b, \alpha; c; \mathbf{I}_m) &= \frac{1}{\beta_m[a, c-a]} \int_{\mathbf{0} < \mathbf{R} < \mathbf{I}_m} |\mathbf{R}|^{c-b-(m+1)/2} |\mathbf{I}_m - \mathbf{R}|^{b-\alpha-(m+1)/2} (d\mathbf{R}) \\ &= \frac{\beta_m[b-\alpha, c-b]}{\beta_m[c-b, b]}. \end{aligned}$$

And from the Kummer relation discussed in Muirhead (1982, Theorem 7.4.3, p. 265 and Theorem 7.3.4), we have

$$\begin{aligned} M_g(\mathbf{Y}) &= \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} {}_1F_1(b; c; -\mathbf{Y})(d\mathbf{Y}) \\ &= \int_{\mathbf{Y} > \mathbf{0}} |\mathbf{Y}|^{\alpha-(m+1)/2} {}_1\text{etr}(-\mathbf{Y}) {}_1F_1(c-b; c; \mathbf{Y})(d\mathbf{Y}) \\ &= \Gamma_m[\alpha] {}_2F_1(c-b, \alpha; c; \mathbf{I}_m) \\ &= \frac{\Gamma_m[\alpha]\Gamma_m[c]\Gamma_m[b-\alpha]}{\Gamma_m[b]\Gamma_m[c-\alpha]}. \quad \square \end{aligned}$$

Now, from Lemmas 2.4 and 2.5 taking  $\mathbf{Y} = \mathbf{\Xi}^{1/2}\mathbf{Y}\mathbf{\Xi}^{1/2}$  with  $(d\mathbf{Y}) = |\mathbf{\Xi}|^{(m+1)/2}(d\mathbf{Y})$ , we obtain the following.

**Definition 2.2.** Let  $\mathbf{\Xi}$  be any arbitrary complex symmetric  $m \times m$  matrix.  $\mathbf{Y}$  is said to have a **matrix variate generalised hypergeometric distribution** if,

1. Its density function is

$$f_{\mathbf{Y}}(\mathbf{Y}) = \frac{|\mathbf{\Xi}|^{\alpha}\beta_m[a, c-a]}{\beta_m[\alpha, b-\alpha]\beta_m[a-\alpha, c-\alpha]} |\mathbf{Y}|^{\alpha-(m+1)/2} {}_2F_1(a, b; c; -\mathbf{\Xi}\mathbf{Y}), \quad \mathbf{Y} > \mathbf{0},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(a-\alpha) > (m-1)/2$ ,  $\text{Re}(b-\alpha) > (m-1)/2$ ,  $\text{Re}(c-\alpha) > (m-1)/2$  and  $\text{Re}(c-a) > (m-1)/2$ .

2. Or

$$f_{\mathbf{Y}}(\mathbf{Y}) = \frac{|\Xi|^\alpha \Gamma_m[b] \Gamma_m[c - \alpha]}{\Gamma_m[\alpha] \Gamma_m[c] \Gamma_m[b - \alpha]} |\mathbf{Y}|^{\alpha - (m+1)/2} {}_1F_1(b; c; -\Xi \mathbf{Y}), \quad \mathbf{Y} > \mathbf{0},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(b - \alpha) > (m-1)/2$  and  $\text{Re}(c - \alpha) > (m-1)/2$ .

**Lemma 2.6.** *Let  $\Xi$  be any arbitrary complex symmetric  $m \times m$  matrix. It is said that  $\mathbf{P} = \mathbf{Y}^{-1}$  has a **matrix variate inverted generalised hypergeometric distribution** if,*

1. *Its density function is*

$$f_{\mathbf{P}}(\mathbf{P}) = \frac{|\Xi|^\alpha \beta_m[a, c - \alpha]}{\beta_m[\alpha, b - \alpha] \beta_m[a - \alpha, c - \alpha]} |\mathbf{P}|^{-\alpha - (m+1)/2} {}_2F_1(a, b; c; -\Xi \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(a - \alpha) > (m-1)/2$ ,  $\text{Re}(b - \alpha) > (m-1)/2$ ,  $\text{Re}(c - \alpha) > (m-1)/2$  and  $\text{Re}(c - a) > (m-1)/2$ .

2. Or

$$f_{\mathbf{P}}(\mathbf{P}) = \frac{|\Xi|^\alpha \Gamma_m[b] \Gamma_m[c - \alpha]}{\Gamma_m[\alpha] \Gamma_m[c] \Gamma_m[b - \alpha]} |\mathbf{P}|^{-\alpha - (m+1)/2} {}_1F_1(b; c; -\Xi \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(b - \alpha) > (m-1)/2$  and  $\text{Re}(c - \alpha) > (m-1)/2$ .

*Proof.* Follows from Definition 2.2, taking  $\mathbf{P} = \mathbf{Y}^{-1}$  with  $(d\mathbf{Y}) = |\mathbf{P}|^{-(m+1)}(d\mathbf{P})$ .  $\square$

### 3 Compound elliptical distribution of a random matrix

In this section we propose several families of elliptical distributions based on an extension to the matrix variate case of the vector case idea, introduced by Muirhead (1982), the approach Known as compound distribution. This approach was used by Roux (1971) and Van der Morwe and Roux (1974) for the distribution of a positive definite random matrix.

In general, assume that the conditional distribution of

$$\mathbf{X}|\mathbf{P} \sim \mathcal{EL}\mathcal{S}_{m \times n}(\boldsymbol{\mu}, \boldsymbol{\Sigma}^{1/2} \boldsymbol{\Psi}(\mathbf{P}) \boldsymbol{\Sigma}^{1/2}, \boldsymbol{\Theta}, h), \quad (2)$$

with  $\boldsymbol{\Psi} : \mathbb{R}^{m(m+1)/2} \rightarrow \mathbb{R}^{m(m+1)/2}$ ,  $\boldsymbol{\Psi}(\mathbf{P}) > \mathbf{0}$ ; where  $\mathbf{P} > \mathbf{0}$  has the distribution function  $G(\mathbf{P})$ . Then  $\mathbf{X}$  has a left-spherical elliptical distribution (compound distribution) with a density function given by

$$\frac{c(m, n)}{|\boldsymbol{\Sigma}|^{n/2} |\boldsymbol{\Theta}|^{m/2}} \int_{\mathbf{P} > \mathbf{0}} \frac{h\left(\boldsymbol{\Psi}(\mathbf{P})^{-1} \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1/2}\right) dG(\mathbf{P})}{|\boldsymbol{\Psi}(\mathbf{P})|^{n/2}}, \quad (3)$$

where  $\boldsymbol{\Psi}(\mathbf{P})^{-1}$  denotes the inverse of the matrix  $\boldsymbol{\Psi}(\mathbf{P})$  (not the inverted function of  $\boldsymbol{\Psi}(\cdot)$ ).

Let us now consider two particular matrix variate left-spherical elliptical distributions, the matrix variate normal and the matrix variate  $T$  distributions, see Dickey (1967), Box and Tiao (1972, pp. 441-448) and Press (1982, pp. 138-141).

### 4 Compound matrix variate normal distribution

Recall that if  $\mathbf{X} \sim \mathcal{N}_{m \times n}(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$ , its density function is given by

$$\frac{1}{(2\pi)^{mn/2} |\boldsymbol{\Sigma}|^{n/2} |\boldsymbol{\Theta}|^{m/2}} \text{etr} \left\{ -\frac{1}{2} \boldsymbol{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \right\}.$$

**Theorem 4.1.** Assume that  $\mathbf{X}|\mathbf{P}$  has a matrix variate normal distribution,

$$\mathbf{X}|\mathbf{P} \sim \mathcal{N}_{m \times n}(\boldsymbol{\mu}, \boldsymbol{\Sigma}^{1/2} \mathbf{P} \boldsymbol{\Sigma}^{1/2}, \boldsymbol{\Theta}),$$

where  $\mathbf{P}$  has a matrix variate inverted hypergeometric gamma type distribution. By Lemma 2.2 its density function is

$$g_{\mathbf{P}}(\mathbf{P}) \propto \text{etr}\{-\boldsymbol{\Xi} \mathbf{P}^{-1}\} |\mathbf{P}|^{-a-(m+1)/2} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \boldsymbol{\Upsilon} \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{|\boldsymbol{\Xi}|^a}{\Gamma_m[a] {}_{p+1}F_q(a_1, \dots, a_p, a; b_1, \dots, b_q; \boldsymbol{\Upsilon} \boldsymbol{\Xi}^{-1})},$$

and  $\text{Re}(a) > (m-1)/2$ . Then  $\mathbf{X}$  has a matrix variate left-spherical elliptical distribution with density function

$$\propto \frac{{}_{p+1}F_q\left(a_1, \dots, a_p, a + \frac{n}{2}; b_1, \dots, b_q; \boldsymbol{\Upsilon} \left(\boldsymbol{\Xi} + \frac{1}{2} \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1/2}\right)^{-1}\right)}{|\boldsymbol{\Xi} + \frac{1}{2} \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1/2}|^{a+n/2}}$$

with constant of proportionality

$$\frac{\Gamma_m[a + n/2] |\boldsymbol{\Xi}|^a}{(2\pi)^{mn/2} \Gamma_m[a] |\boldsymbol{\Sigma}|^{n/2} |\boldsymbol{\Theta}|^{m/2} {}_{p+1}F_q(a_1, \dots, a_p, a; b_1, \dots, b_q; \boldsymbol{\Upsilon} \boldsymbol{\Xi}^{-1})}.$$

where  $\text{Re}(a) > (m-1)/2$ .

*Proof.* Follows immediately from 3 and Lemma 2.2.  $\square$

Observe that, by taking  $\boldsymbol{\Upsilon} = \mathbf{0}$  in Theorem 4.1 we obtain that  $\mathbf{X}$  has a matrix variate  $T$  distribution, see Dickey (1967), Box and Tiao (1972, pp. 441-448) and Press (1982, pp. 138-141). Also, if we take  $p = 0$ ,  $q = 1$  and  $a = b_1$  we obtain that  $\mathbf{X}$  has a noncentral matrix variate  $T$  type 2 distribution. Then observing that

$${}_1F_1(a; a; \boldsymbol{\Upsilon} \boldsymbol{\Xi}^{-1}) = \text{etr}\{\boldsymbol{\Upsilon} \boldsymbol{\Xi}^{-1}\}$$

and by the Kummer relation (Muirhead, 1982, eq. (6), p. 265)

$$\begin{aligned} & {}_1F_1\left(a + \frac{n}{2}; a; \boldsymbol{\Upsilon} \left(\boldsymbol{\Xi} + \frac{1}{2} \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1/2}\right)^{-1}\right) \\ &= \text{etr}\left\{\boldsymbol{\Upsilon} \left(\boldsymbol{\Xi} + \frac{1}{2} \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1/2}\right)^{-1}\right\} \\ & \quad \times {}_1F_1\left(-\frac{n}{2}; a; -\boldsymbol{\Upsilon} \left(\boldsymbol{\Xi} + \frac{1}{2} \boldsymbol{\Sigma}^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \boldsymbol{\Sigma}^{-1/2}\right)^{-1}\right). \end{aligned}$$

Observe that for  $n/2$  an integer,  ${}_1F_1$  is a polynomial of degree  $mn/2$ . In this case the density of  $\mathbf{X}$  is evaluated easily, see Muirhead (1982, p. 258).

From (3) and Lemma 2.3 we have the following result.

**Theorem 4.2.** Assume that  $\mathbf{X}|\mathbf{P} \sim \mathcal{N}_{m \times n}(\boldsymbol{\mu}, \boldsymbol{\Sigma}^{1/2} \mathbf{P} \boldsymbol{\Sigma}^{1/2}, \boldsymbol{\Theta})$ , where  $\mathbf{P}$  has a matrix variate inverted hypergeometric beta type II distribution. From Lemma 2.3, its density function is

$$g_{\mathbf{P}}(\mathbf{P}) \propto |\mathbf{P}|^{b-(m+1)/2} |\mathbf{I}_m + \mathbf{P}|^{-(a+b)} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \boldsymbol{\Xi} \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$



where the constant of proportionality is

$$\frac{1}{\beta_m[a, b]_{p+1} F_{q+1}(a_1, \dots, a_p, a; b_1, \dots, b_q, -b + (m+1)/2; -\Xi)},$$

and  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2 + k_1$ . Then  $\mathbf{X}$  has a matrix variate left-spherical elliptical distribution with density function

$$\propto {}_{p+1}F_{q+1} \left( a_1, \dots, a_p, a + n/2; b_1, \dots, b_q, -b + \frac{(m+1)}{2}; \right. \\ \left. -\Xi + \frac{1}{2} \Sigma^{-1/2} (\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \Sigma^{-1/2} \right)$$

where the constant of proportionality is

$$\frac{(2\pi)^{-mn/2} \beta_m[a + n/2, b - n/2] |\Sigma|^{-n/2} |\boldsymbol{\Theta}|^{-m/2}}{\beta_m[a, b]_{p+1} F_{q+1}(a_1, \dots, a_p, a + n/2; b_1, \dots, b_q, -b + (m+n+1)/2; -\Xi)},$$

where  $\text{Re}(a) > (m-1)/2$ ,  $\text{Re}(b) > (m+n-1)/2 + k_1$ .

A result of particular interest is obtained from Theorem 4.2 taking  $\Xi = \mathbf{0}$ . Similarly, in the Bayesian context, Theorem 4.2 generalises a result given in Xu (1990), which can be obtained by taking  $\Xi = \mathbf{0}$  and  $p = q = 0$ . In this latter case, by applying the Kummer relation (Muirhead, 1982, Theorem 7.4.3, p. 265) we obtain a matrix variate confluent hypergeometric of the first kind distribution type.

**Theorem 4.3.** Assume that  $\mathbf{X}|\mathbf{P} \sim \mathcal{N}_{m \times n}(\boldsymbol{\mu}, \Sigma^{1/2} \mathbf{P} \Sigma^{1/2}, \boldsymbol{\Theta})$ , where  $\mathbf{P}$  has a matrix variate inverted generalised hypergeometric distribution. By Lemma 2.6,

1. its density function is,

$$g_{\mathbf{P}}(\mathbf{P}) \propto |\mathbf{P}|^{-\alpha-(m+1)/2} {}_2F_1(a, b; c; -\Xi \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{|\Xi|^\alpha \beta_m[a, c-a]}{\beta_m[\alpha, b-\alpha] \beta_m[a-\alpha, c-\alpha]},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(a-\alpha) > (m-1)/2$ ,  $\text{Re}(b-\alpha) > (m-1)/2$ ,  $\text{Re}(c-\alpha) > (m-1)/2$  and  $\text{Re}(c-a) > (m-1)/2$ .

2. Or with density function

$$g_{\mathbf{P}}(\mathbf{P}) \propto |\mathbf{P}|^{-\alpha-(m+1)/2} {}_1F_1(b; c; -\Xi \mathbf{P}^{-1}), \quad \mathbf{P} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{|\Xi|^\alpha \Gamma_m[b] \Gamma_m[c-\alpha]}{\Gamma_m[\alpha] \Gamma_m[c] \Gamma_m[b-\alpha]},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(b-\alpha) > (m-1)/2$  and  $\text{Re}(c-\alpha) > (m-1)/2$ .

Then  $\mathbf{X}$  has a matrix variate left-spherical elliptical distribution and

1. its density function is

$$\propto |\Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1}(\mathbf{X} - \boldsymbol{\mu})|^{-(\alpha+n/2)} \\ \times {}_3F_1 \left( a, b, \alpha + n/2; c; -2\Xi \left( \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \Sigma^{-1/2} \right)^{-1} \right),$$

where the constant of proportionality is

$$\frac{2^{m\alpha} |\Xi|^\alpha \Gamma_m[\alpha + n/2] \beta_m[a, c - a]}{\pi^{mn/2} \beta_m[\alpha, b - \alpha] \beta_m[a - \alpha, c - \alpha] |\Sigma|^{n/2} |\boldsymbol{\Theta}|^{m/2}},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(a - \alpha) > (m-1)/2$ ,  $\text{Re}(b - \alpha) > (m-1)/2$ ,  $\text{Re}(c - \alpha) > (m-1)/2$  and  $\text{Re}(c - a) > (m-1)/2$ .

2. Or with density function given by

$$\propto |\Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1}(\mathbf{X} - \boldsymbol{\mu})|^{-(\alpha+n/2)} \\ \times {}_2F_1 \left( a, \alpha + n/2; b; -2\Xi \left( \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \Sigma^{-1/2} \right)^{-1} \right),$$

where the constant of proportionality is

$$\frac{2^{m\alpha} |\Xi|^\alpha \Gamma_m[\alpha + n/2] |\Xi|^\alpha \Gamma_m[b] \Gamma_m[c - \alpha]}{\pi^{mn/2} \Gamma_m[\alpha] \Gamma_m[c] \Gamma_m[b - \alpha] |\Sigma|^{n/2} |\boldsymbol{\Theta}|^{m/2}},$$

with  $\text{Re}(\alpha) > (m-1)/2$ ,  $\text{Re}(b - \alpha) > (m-1)/2$  and  $\text{Re}(c - \alpha) > (m-1)/2$ .

*Proof.* Follows from (3) and Lemma 2.6.  $\square$

## 5 Compound matrix-variate $T$ distribution

From Dickey (1967), Box and Tiao (1972, pp. 441-448) and Press (1982, pp. 138-141) we know that  $\mathbf{X}$  has a matrix-variate  $T$  distribution, denoting this fact as  $\mathbf{X} \sim \mathcal{MT}_{m \times n}(\nu, \boldsymbol{\mu}, \Sigma, \boldsymbol{\Theta})$ , if its density function is

$$\frac{\Gamma_m[(n + \nu)/2]}{\pi^{mn/2} \Gamma_m[\nu/2] |\Sigma|^{n/2} |\boldsymbol{\Theta}|^{m/2}} |\mathbf{I}_m + \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1}(\mathbf{X} - \boldsymbol{\mu})|^{-(n+\nu)/2}.$$

where  $\nu > m - 1$ .

**Theorem 5.1.** Assume that  $\mathbf{X}|\mathbf{P} \sim \mathcal{MT}_{m \times n}(\nu, \boldsymbol{\mu}, \Sigma^{1/2} \mathbf{P} \Sigma^{1/2}, \boldsymbol{\Theta})$ , where  $\mathbf{P}$  has a matrix-variate inverted hypergeometric beta type II distribution with  $\Xi = \mathbf{0}$ . From Lemma 2.3, its density function is

$$g_{\mathbf{P}}(\mathbf{P}) \propto |\mathbf{P}|^{b-(m+1)/2} |\mathbf{I}_m + \mathbf{P}|^{-(a+b)}, \quad \mathbf{P} > \mathbf{0},$$

where the constant of proportionality is

$$\frac{1}{\beta_m[a, b]},$$

and  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m-1)/2$ . Then  $\mathbf{X}$  has a matrix-variate left-spherical elliptical distribution with density function

$$\propto {}_2F_1 \left( \frac{(n + \nu)}{2}, a + \frac{n}{2}; -b + \frac{(m + n + 1)}{2}; \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \boldsymbol{\Theta}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \right)$$

where the constant of proportionality is

$$\frac{\Gamma_m[(n+\nu)/2]\beta_m[a+n/2, b-n/2]}{\pi^{mn/2}\Gamma_m[\nu/2]\beta_m[a, b]|\Sigma|^{n/2}|\Theta|^{m/2}},$$

where  $\text{Re}(a) > (m-1)/2$  and  $\text{Re}(b) > (m+n-1)/2 + k_1$ .

*Proof.* Follows from Lemma 2.3, noting that

$$\begin{aligned} & \left| \mathbf{I}_m + \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})' \Theta^{-1}(\mathbf{X} - \boldsymbol{\mu}) \Sigma^{-1/2} \mathbf{P}^{-1} \right|^{-(n+\nu)/2} \\ &= {}_1F_0 \left( \frac{(n+\nu)}{2}; -\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})' \Theta^{-1}(\mathbf{X} - \boldsymbol{\mu}) \Sigma^{-1/2} \mathbf{P}^{-1} \right). \quad \square \end{aligned}$$

By applying the Euler relation (Muirhead, 1982, eq. (7), p. 265) to results in Theorem 5.1, the density function of  $\mathbf{X}$  is then

$$\begin{aligned} & \propto |\Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \Theta^{-1}(\mathbf{X} - \boldsymbol{\mu})|^{-(a+b+(n+\nu)/2-(m+1)/2)} {}_2F_1 \left( -b - \frac{\nu}{2} + \frac{m+1}{2}, \right. \\ & \quad \left. -a-b + \frac{m+1}{2}; -b + \frac{(m+n+1)}{2}; \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \Theta^{-1}(\mathbf{X} - \boldsymbol{\mu}) \right). \end{aligned}$$

where, if  $a$  and  $b$  are integers,  $2(a+b) > m+1$  and  $m$  is odd,  ${}_2F_1$  is a polynomial of degree  $m(a+b-(m+1)/2)$ . Similarly, if  $b$  and  $\nu/2$  are integers,  $2(a+\nu/2) > m+1$  and  $m$  is odd,  ${}_2F_1$  is a polynomial of degree  $m(a+\nu/2-(m+1)/2)$ , see Muirhead (1982, p. 258).

## 6 Scale mixture of elliptical distribution of a random matrix

The approach known as the scale mixture of normal distributions, proposed by Muirhead (1982, p. 33) for the vector case and extended by Gupta and Varga (1993, Chapter 4) to the matrix variate case, is obtained as a particular case of the approach described in the Section 3. To do so, we take  $m = 1$  in the distribution  $G(\mathbf{P})$ , from where we obtain the following approach, termed the scale mixture of an elliptical distribution, cited by Gupta and Varga (1993, pp. 78–79).

Assume that the conditional distribution

$$\mathbf{X}|s \sim \mathcal{EL}\mathcal{S}_{m \times n}(\boldsymbol{\mu}, \phi(s)\Sigma, \Theta, h), \quad (4)$$

where  $\phi : (0, \infty) \rightarrow (0, \infty)$ , with  $s > 0$  has the distribution function  $G(s)$ . Then  $\mathbf{X}$  has a left-spherical elliptical distribution (scale mixture of elliptical distribution) with a density function given by

$$\frac{c(m, n)}{|\Sigma|^{n/2}|\Theta|^{m/2}} \int_{s>0} (\phi(s))^{-mn/2} h \left( \frac{1}{\phi(s)} \Sigma^{-1}(\mathbf{X} - \boldsymbol{\mu})' \Theta^{-1}(\mathbf{X} - \boldsymbol{\mu}) \right) dG(s). \quad (5)$$

As an example, consider the following version of Theorem 2.6 for  $m = 1$ .

**Theorem 6.1.** Assume that  $\mathbf{X}|s$  has a matrix variate normal distribution,

$$\mathbf{X}|s \sim \mathcal{N}_{m \times n}(\boldsymbol{\mu}, s\Sigma, \Theta),$$

where  $s$  has an inverted hypergeometric gamma type distribution. By Lemma 2.2 its density function is

$$g_s(s) \propto \exp \left\{ -\frac{\xi}{s} \right\} s^{-a-1} {}_pF_q \left( a_1, \dots, a_p; b_1, \dots, b_q; \frac{v}{s} \right), \quad s > 0,$$

where  $v > 0$ ,  $\xi > 0$  and the constant of proportionality is

$$\frac{\xi^a}{\Gamma[a]_{p+1} F_q \left( a_1, \dots, a_p, a; b_1, \dots, b_q; \frac{v}{\xi} \right)},$$

and  $\text{Re}(a) > 0$ . Then  $\mathbf{X}$  has a matrix variate left-spherical elliptical distribution with density function

$$\propto \frac{{}_{p+1}F_q \left( a_1, \dots, a_p, a + \frac{mn}{2}; b_1, \dots, b_q; v \left( \xi + \frac{1}{2} \text{tr} \mathbf{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})' \mathbf{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \right)^{-1} \right)}{\left( \xi + \frac{1}{2} \text{tr} \mathbf{\Sigma}^{-1} (\mathbf{X} - \boldsymbol{\mu})' \mathbf{\Theta}^{-1} (\mathbf{X} - \boldsymbol{\mu}) \right)^{a+mn/2}}$$

with constant of proportionality

$$\frac{\Gamma[a + mn/2] \xi^a}{(2\pi)^{mn/2} \Gamma[a] |\mathbf{\Sigma}|^{n/2} |\mathbf{\Theta}|^{m/2} {}_{p+1}F_q \left( a_1, \dots, a_p, a; b_1, \dots, b_q; \frac{v}{\xi} \right)}.$$

where  $\text{Re}(a) > 0$ .

Similarly, from Theorem 6.1 particular cases are obtained, taking, for example,  $v = 0$ , but in this case we obtaining the matrix variate  $T$  distribution (not the matricvariate  $T$  distribution). Similar consequences are obtained as particular cases, taking  $m = 1$  from the Theorems 4.2-5.1.

## Conclusions

This paper introduces several hypergeometric type distributions, which include many that are well-known in the statistical literature, such as central and noncentral matrix variate inverted Gamma (Wishart) and matrix variate beta type II distributions, among many others.

Assuming that  $\mathbf{P}$  has one of these hypergeometric distributions in

$$\mathbf{X}|\mathbf{P} \sim \mathcal{E}\mathcal{L}\mathcal{S}_{m \times n}(\boldsymbol{\mu}, \mathbf{\Sigma}^{1/2} \boldsymbol{\Psi}(\mathbf{P}) \mathbf{\Sigma}^{1/2}, \mathbf{\Theta}, h),$$

several left-spherical matrix variate elliptical families are found. These enable us to study examples in which the tails of the distributions are heavier or lighter than in the normal case. These approaches to obtaining elliptical distributions are of particular interest from the Bayesian standpoint and for shape theory distributions, see Jammalamadaka *et al.* (1987) and Caro-Lopera *et al.* (2008), respectively.

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